Barisal University Journal Part 1, 4(2):325-338 (2017) ISSN 2411-247X

APPLICATION OF FIXED POINT THEOREM OF MEASURE OF WEAK NONCOMPACTNESS TO NONLINEAR VOLTERRA-HAMMERSTEIN INTEGRAL EQUATIONS

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Abstract

In this paper some fixed point theorems and their generalization for the sum of two operators has been presented. For that purpose the measure of weak noncompactness and sequentially weakly continuous mappings and the Darbo-Sadovskii fixed point theorem are used. Finally an application to Volterra- Hammerstein integral equations is given to illustrate the applicability of our result.

Keywords**:** Fixed point theorem, measure of noncompactness, condensing mappings, Volterra-Hammerstein integral equations.

Introduction

Last few decades the study of differential equations, integral equations and integral-differential equations related to fixed point theorem has been an enthusiastic area of research. The theory of Fredholm, Volterra and Hammerstein integral equations has undergone rapid development. In this paper we have represented the existence of solutions for Volterra-Hammerstein integral equation in view of measure of noncompactness. That's why the measure of noncompactness arises due to some difficulties in compactness which stated as every continuous function from a convex compact subset *K* of a Banach space to *K* itself has a fixed point by Schauder. This fixed point theorem was one of the most efficient tools in nonlinear functional analysis to solve differential and integral equations. When compactness seems difficult in several situations, the Tychonov fixed point theorem appears as a good alternative which asserts that every weakly continuous and weakly compact mapping from a nonempty closed convex subset of a Banach space to itself has a fixed point. Later De Blasi (1977) introduce the concept of measure of weak noncompactness. Since then several fixed point theorem have been proved like Darbo type, Sadovaskii type, Krasnosel'skii type and

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many others for sequentially weakly continuous mappings. More recently a generalized version of Sadovskii's fixed point theorem for sequentially weakly continuous mappings has been proved (Agarwal, et al. O'Regan and Taoudi, 2012). Using the concept, a new class of mappings in Banach space was defined named convex power condensing. More generally, with help of that measure of noncompactness an existence result for a nonlinear quadratic integral equation of Hammerstein type in Holder spaces have been proved (Banas and Nalepa, 2016). Also, (Allahyari, et al., 2013**)** proved the existence of solutions for some classes of integro-differential equations.

The outline of this paper consists of preliminaries in section 1 where mentioned important definitions, lemmas and theorems related to measure of noncompactness. Section 2 is the main result where the concept on Volterra-Hammerstein integral equations was applied to justify the accountability of result obtained.

1. Preliminaries

Let $(B, \| \ \|)$ be a Banach space with the zero element θ. Denote the collection of all nonempty bounded subsets of B by $\mathfrak{M}_B$ and the subset of $\mathfrak{M}_B$ consisting of all weakly compact subsets of B by $\mathfrak{N}_B$. Further let $\bar{A}$ and $Conv\,(A)$ denote the closure and the convex hull of A respectively. Let B_R denote the closed ball in E centered at 0 with radius $r > 0$. The De Blasi measure of weak noncompactness is the map $\gamma: \mathfrak{M}_B \to [0, \infty)$ defined by

$$\gamma(A) = \inf\{r > 0: there\ exists\ a\ set\ \gamma \in \mathfrak{N}_B\ such\ that\ A \subseteq \gamma + \mathrm{B}_R\} \text{ for all } A \in \mathfrak{M}_B.$$

Definition 1.1 (Bana´s and Goebel, 1980) A mapping $\gamma: \mathfrak{M}_B \to [0, \infty)$ is said to be a measure of noncompactness in B if the following condition are satisfied:

(i) The family $\ker \gamma = \{X \in \mathfrak{M}_B : \gamma(X) = 0\}$ is nonempty and $\ker \gamma \subset \mathfrak{N}_B$.

(ii) $\gamma(A) = \gamma(\overline{X}^{\gamma})$, where $\overline{X}^{\gamma}$ is the weak closure of X;

(iii) $\gamma\big(co(X)\big) = \gamma(X)$, where; $co(X)$ denotes the convex hull of X

(iv) If $X \subset Y$, then $\gamma(X) \leq \gamma(Y)$;

(v) $\gamma(X \cup Y) = \max\{\gamma(X), \gamma(Y)\}$;

(vi) $\gamma(\lambda X) = |\lambda| \gamma(X)\ for\ \lambda \in \mathbb{R}$, where $\lambda X = \{\lambda x: x \in X\}$;

(vii) $\gamma(X + Y) \leq \gamma(X) + \gamma(Y)$, where $X + Y = \{x + y: x \in X, y \in Y\}$;

(viii) If X_n is a sequence of nonempty, weakly closed subsets of B with X_1 bounded and $X_1 \supseteq X_2 \supseteq X_3 \dots \dots \dots \supseteq X_n \dots$ with $\lim_{n\to\infty} \gamma(X_n) = 0$, then $\bigcap_{n=1}^{\infty} X_n \neq \emptyset$ and $\gamma(\bigcap_{n=1}^{\infty} X_n) = 0$.

(ix) For all $\lambda \in [0,1]$,

$$\gamma(\lambda X + (1-\lambda)Y) \leq \lambda\gamma(X) + (1-\lambda)\gamma(Y).$$

The family $\ker\gamma$ described in (i) is said to be the kernel of the measure of noncompactness γ.

By a measure of weak noncompactness, we mean a map $\mu: \Omega_E \to \mathbb{R}^+$ satisfying the properties (i)-(ix) quoted above. In what follows, we need the following definition. Let B be a Banach space, let M be a nonempty closed convex subset of B, and let $S, T: M \to B$ be two nonlinear mappings and $x_0 \in B$. For any $N \subseteq M$, we get

$$F^{(1,x_0)}(T,S,N) = \{x \in M: x = Sx + Ty \; for \; some \; y \in N\}$$

and

$$F^{(n,x_0)}(T,S,N) = F^{(1,x_0)}\left(T,S,\overline{co}(F^{(n-1,x_0)}(T,S,N) \cup \{x_0\})\right)$$

for $n = 2,3,\dots\dots$

Definition 1.2 (Hussain and Taoudi, 2013) Let B be a Banach space, let M be a nonempty closed convex subset of B, and let μ be a measure of weak noncompactness on B. Let $T, S: M \to B$ be two bounded mappings (i.e. they take bounded sets into bounded ones) and $x_0 \in M$. We say that T is an S-convex-power condensing operator about x_0 and n_0 with respect to μ if for any bounded set $N \subseteq M$ with $\mu(N) > 0$ we have

$$\mu\left(F^{(n,x_0)}(T,S,N)\right) < \mu(N)$$

Obviously, $T: M \to M$ is power-convex condensing with respect to μ about x_0 and n_0 if and only if it is a 0-convex-power condensing operator about x_0 and n_0 w.r.t. μ.

Definition 1.3 (Hussain and Taoudi, 2013) A mapping $D(T) \subset X, T: D(T) \to X$ is called k- Lipschitzian if $\|Tx - Ty\| \leq k\|x - y\|$ for all $x, y \in D(T)$. T is called strict contraction if $k \in [0,1)$ and nonexpansive if $k = 1$.

Theorem 1.1 (Mitchell and Smith, 1978) Let B be a Banach space and let $H \subseteq C([0,T],B)$ be bounded and equicontinuous. Then the map $t \to \gamma(H(t))$ is continuous on $[0,T]$ and

$$\gamma(H) = sup_{t\in[0,T]}\gamma\left(H(t)\right) = \gamma(H[0,T]),$$

where $H(t) = \{h(t): h \in H\}$ and $H[0,T] = \bigcup_{t\in[0,T]}\{h(t): h \in H\}$.

Theorem 1.2 (Dobrakov, 1971) Let S be a Hausdroff compact space and B be a Banach space. A bounded sequence $(f_n) \subset C(S,B)$ converges weakly to $f \in C(S,B)$ if and only if, for every $t \in S$, the sequence $(f_n(t))$ converges weakly $(in\ B)$ to $f(t)$.

Theorem 1.3 (Aghajani et al., 2013) Let M be a nonempty, bounded, closed, and convex subset of a Banach space B and let $T: M \to M$ be a continuous function satisfying $\gamma\big(T\ (W\ )\big) \leq \varphi\big(\gamma(W\ )\big)$ for each $W \subset M$, where γ is an arbitrary measure of noncompactness and $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is a monotone increasing (not necessarily continuous) function with

$\lim_{n\to\infty} \varphi^n(t) = 0 \ \ for\ all \quad t \geq 0.$

Then T has at least one fixed point in M.

Theorem 1.4 (Hussain and Taoudi, 2013) Let M be a nonempty bounded closed convex subset of Banach space B, and let μ be a measure of weak noncompactness on B. Suppose that $T: M \to B$ and $S: B \to B$ are two mappings satisfying

(i) T is sequentially weakly continuous

(ii) S is strict contraction

(iii) There are an integer n_0 and a vector $x_0 \in B$ such that T is S-power-convex condensing with respect to μ about x_0 and n_0

(iv) If $x = Sx + Ty$, for some $y \in M$, then $x \in M$

(v) If $\{x_n\}$ is a sequence in $F^{(n_0,x_0)}(T,S,M)$ such that $x_n \rightharpoonup x$, then $Sx_n \rightharpoonup Sx$. Then $T + S$ has at least one fixed point in M.

It is worthwhile to emphasize that theorem 1.4 encompasses a lot of previously known results. In particular, if we take $B = 0$ in theorem 1.4, we recapture the following fixed point theorem, which was proved (Agarwal et al., 2012).

Corollary 1.1 Let M be a nonempty bounded closed convex subset of a Banach space X. Suppose that $T: M \to M$ is weakly sequentially continuous and there exist an integer n_0 and a vector $x_0 \in B$ such that T is power-convex condensing about x_0 and n_0. Then T has at least one fixed point in M**.**

Another consequence of theorem 1.4 is the following result, which is a sharpening of (Barroso, 2005).

Corollary 1.2 Let M be a nonempty bounded closed convex subset of a Banach space B. Suppose that $T: M \to B$ and $S: B \to B$ are two mappings satisfying:

(i) T is sequentially weakly continuous

(ii) S is strict contraction with constant k

(iii) There exists an integer n_0 such that $F^{(n_0,x_0)}(T,S,M)$ is relatively weakly compact

(iv) If $x = Sx + Ty$, for some $y \in M$, then $x \in M$

(v) If $\{x_n\}$ is a sequence in $F^{(n_0,x_0)}(T,S,M)$ such that $x_n \rightharpoonup x$, then $Sx_n \rightharpoonup Sx$.

Then $T + S$ has at least one fixed point in M.

In order to state another consequence of Theorem 1.4, the following abstract lemma is very useful.

Lemma 1.1 (Hussain and Taoudi, 2013) Assume that the conditions (i), (ii) and (iv) of Theorem 1.4 hold. If, moreover, S is sequentially weakly continuous and $T(M)$ is relatively weakly compact, then the set $F \coloneqq F^{(1,x_0)}(T,S,M) \coloneqq \{x \in M: x = Sx + Ty \; for\ some\ \ y \in M\}$ is relatively weakly compact.

Proof From the definition of F it follows that

$$F \subset S(F) + T(M).$$

Keeping in mind that $T(M)$ is relatively weakly compact, we get

$$w(F) \leq w\big(S(F)\big) + w(T(M))$$

$$= w(S(F))$$

$$\leq kw(F).$$

Since $0 \leq k < 1$, then $w(F) = 0$ and therefore F is relatively compact.

On the basis of Lemma 1.1, the following Krasnosel'skii-type fixed point theorem follows from Theorem 1.4.

Corollary 1.3 (Taoudi, 2010) Let M be a nonempty bounded closed convex subset of a Banach space B. Suppose that $T: M \to B$ and $S: B \to B$ are two sequentially weakly continuous mappings satisfying:

(i) $T(M)$ is relatively weakly compact

(ii) S is strict contraction

(iii) If $x = Sx + Ty$, for some $y \in M$, then $x \in M$

Then $T + S$ has at least one fixed point in M.

In some applications, the sequential weak continuity condition is not easy to be verified. We thus consider the following two conditions: let $T: D(T) \subset E \to E$ be a map.

(H1) f $(x_n)_{n\in\mathbb{N}}$ is a weakly convergent sequence in $D(T)$, then $(x_n)_{n\in\mathbb{N}}$ has a strongly convergent sequence in B.

(H2) If $(x_n)_{n\in\mathbb{N}}$ is a weakly convergent sequence in $D(T)$, then $(x_n)_{n\in\mathbb{N}}$ has a weakly convergent sequence in B.

Continuous mappings satisfying (H1) are called ws-compact mappings and continuous mappings satisfying (H2) are called ww-compact mappings.

Theorem 1.5 (Hussain and Taoudi, 2013) Let M be a nonempty bounded closed convex subset of a reflexive Banach space B, and let μ be a measure of weak noncompactness on B. Suppose that $T: M \to B$ and $S: B \to B$ are two continuous mappings satisfying:

(i) T verifies (H1)

(ii) S is a strict contraction verifying (H2)

(iii) There are an integer n_0 and a vector $x_0 \in B$ such that T is S-power-convex condensing with respect to μ about n_0 and x_0

(iv) $x = Tx + Sy \in M$ for some $y \in M$ then $x \in M$.

Then $T + S$ has at least one fixed point in M.

2. Main Result

In this section we shall discuss the existence of solutions to the Volterra-Hammerstein integral equation

$$x(t) = f(\ x(t)) + \int_0^t K(t,s) g(s, x(s)) ds, \quad T \in [0,1) \tag{2.1}$$

where $K \in L^1$ is an scalar kernel, B is a Banach space with the norm $\|.\|$ and $g : [0,T] \times B \to B$.

The integral in (2.1) is understood to be the Pettis integral and solutions to (2.1) will be sought in $B := C([0,T], X)$

This equation will be studied under the following assumptions:

(i) For each $t \in [0,T], g_t = g(t,.)$ is sequentially weakly continuous (i.e., for each $t \in [0,T]$, for each weakly convergent sequence (x_n), the sequence $g_t(x_n)$ is weakly convergent.).

(ii) There exists at least one solution $r \in C(I,(0,\infty))$ to the inequality

$$\theta(\|r\|_0)\int_0^t K(t,s)\alpha(s)ds \leq r(t), \qquad t \in I$$

where $\|.\|_0$ is the sup norm in $C(I,(0,\infty))$ and $I = [0,1)$.

(iii) g satisfies Carathéodory type conditions; i.e., $g(\cdot,x)$ is measurable for each x and $g(t,\cdot)$ is continuous for a.e. $t \in I$.

(iv) There exist $\alpha \in L^1[0,T]$ and $\theta: [0,+\infty) \to (0,+\infty)$ a nondecreasing continuous function such that $\|g(s,u)\| \leq \alpha(s)\theta(\|u\|)$ for a.e.$s \in [0,T]$

(v) There is a constant $\lambda \geq 0$ such that for any bounded subset S of X and for any $t \in [0,T]$, we have

$$\gamma(g([0,T] \times S)) \leq \lambda\gamma(S)$$

(vi) $f: X \to X$ is sequentially weakly continuous.

(vii) $|K(t,s)| \leq M_\alpha$ where M_α is a convergent type Kernel.

(viii) There exists $k \in [0,1)$ such that $\|f(u) - f(v)\| \leq k\|u - v\|$ for all $u, v \in X$.

Theorem 2.1: Let X be a Banach space and suppose (i)-(viii) hold. Then equation (2.1) has a solution in $C(I,X)$. Let $W: C(I,X) \to C(I,X)$ be defined by

$$(Wx)(t) = \int_0^t K(t,s)g\big(s,x(s)\big)ds \ , \qquad t \in I$$

$\forall\, x \in C(I,B)$. Let $r(t)$ be the function which satisfies condition (ii) and

$r_0 = inf_{t \in r(t)}\, r(t) > 0$.

Define the set, $B_p = \{x \in C(I,B) : \|x\| \leq r_0\}$.

Proof: Let

$M = \{x \in C(I,X): \|x(t)\| \leq r_1(t)$ for $t \in [0,T]$, where $r_1(t) \geq \frac{M_\alpha\, \varphi(\|u(s)\|)}{1-k}$ and

$\|x(t) - x(s)\| \leq r_2(t)$, where $r_2(t) \geq \frac{M_\alpha\, \varphi(\|u(t)-u(s)\|)}{1-k}$ for $t,s \in [0,T]\}$, for all $t \in [0, T]$. Also notice that M is closed, convex, bounded, equicontinuous subset of $C([0, T], X)$ with $0 \in C$. To allow the abstract formulation of equation (2.1), we define the following operators $S, T: C([0, T], X) \to C([0, T], X)$ by

$$(Tx)(t) = x_0 + \int_0^t K(t, s)g\big(s, x(s)\big)ds$$

and

$$(Sx)(t) = f\big(x(t)\big) - x_0.$$

Our strategy is to apply Theorem 1.4 to show the existence of a fixed point for the sum $S + T$ in M which in turn is a continuous solution for equation (2.1). The proof will be divided into several steps.

Step1: First we show that S and T are continuous from $C([0, T], X)$ into itself.

To see this, let x_n be a sequence in $C([0, T], X)$ such that $\lim_{n\to\infty} x_n = x$ in $C([0, T], X)$.

Thus $|Tx_n - Tx| \leq \int_0^t |K(t, s)|\, |g(s, x_n(s) - g(s, x(s)))| ds$

$$\leq M_\alpha \int_0^t |g(s, x_n(s) - g(s, x(s)))| ds \qquad \text{[by (vii)]}$$

Using the dominated convergence theorem, we deduce that, $\lim_{n\to\infty} |Tx_n - Tx| = 0$. The continuity of S follows from (viii).

Step 2: We show that T is equicontinuous. It is enough to prove that T is equicontinuous because the concepts of boundedness and equicontinuity have hereditary properties related to the closure of the convex hull in $C(I, E)$. Indeed, let $t_2 \in T$ and $0 \leq t_1 < t_2 \leq T$ be given. Then

$$\|(Tx)(t_2) - (Tx)(t_1)\| = \left\| \int_0^{t_2} K(t_2, s)g(s, x(s))ds - \int_0^{t_1} K(t_1, s)g(s, x(s))ds \right\|$$

$$\leq \int_0^{t_1} |K(t_2, s) - K(t_1, s)|\, \|g(s, x(x))\| + \int_{t_1}^{t_2} |K(t_2, s)|\, \|g(s, x(x))\| ds$$

$$\leq \int_0^{t_1} |K(t_2, s) - K(t_1, s)|\, \alpha(s)\theta(\|u\|)ds + \int_{t_1}^{t_2} |K(t_2, s)|\, \alpha(s)\theta(\|u\|)ds$$

$$\leq \theta(\|u\|) \int_0^{t_1} |K(t_2, s) - K(t_1, s)|\, \alpha(s)ds + \theta(\|u\|) \int_{t_1}^{t_2} |K(t_2, s)|\, \alpha(s)ds$$

Using Theorem 1.5 and (ii) we deduce that $\||K(t_2, s) - K(t_1, s)| \to 0$ as $t_1 \to t_2$

Since the function $t_2 \to C(I, X)$, then the Lebesgue dominated convergence theorem implies that

$\int_0^{t_1} |K(t_2, s) - K(t_1, s)| \, \alpha(s) ds \to 0$ as $t_1 \to t_2$.
Moreover, $\theta(\|u\|) \int_0^{t_1} |K(t_2, s) - K(t_1, s)| \, \alpha(s) ds \to 0$ as $t_1 \to t_2$.
Consequently, T is equicontinuous on $[0, T]$.

Step 3: Now show that W is bounded from B_p into itself. For $x, y \in B_p$ and $t \in [0, T]$, we have

$$\begin{aligned}\|(Wx)(t)\| &\leq \left\| \int_0^t K(t, s) g\big(s, x(s)\big) ds \right\| \\ &\leq \int_0^t K(t, s) \alpha(s) \theta(\|u\|) ds \\ &\leq \theta(\|r\|_0) \int_0^t K(t, s) \alpha(s) ds \leq r(t) \\ &\leq r(t) \qquad \text{[by (ii)]}\end{aligned}$$

This follows that W is bounded from B_p into itself.

Step 4: Let $x \in C([0, T], X)$ such that $x = Sx + Ty$ with $y \in M$.For all $t \in C[0, T]$ we have

$$x(t) = f\big(x(t)\big) + \int_0^t K(t, s) g\big(s, x(s)\big) ds \tag{2.2}$$

Hence,

$$\begin{aligned}\|x(t)\| &\leq \big\|f\big(x(t)\big)\big\| + \int_0^t K(t, s) \|g(s, x(s))\| ds \\ &\leq \|f(0)\| + k\|x(t)\| + M_\alpha \int_0^t \alpha(s) \theta(\|u\|) ds \\ &\leq k\|x(t)\| + M_\alpha \theta(\|u\|) \int_0^t \alpha(s) ds \\ &\leq k\|x(t)\| + M_\alpha \theta(\|u\|) \int_0^t \alpha(s) ds \\ &\leq \frac{M_\alpha \theta(\|u\|)}{1-k} \int_0^t \alpha(s) ds \\ &\leq r_1\end{aligned}$$

Accordingly,

$$\|x(t)\| \leq r_1$$

On the other hand, let $t, s \in [0, T]$ with $t < s$ and let $x \in C([0, T], X)$. Then

$$\begin{aligned}\|x(t) - x(s)\| &\leq \big\|f\big(x(t)\big) - f\big(x(s)\big)\big\| + \int_t^s \|g(r, x(r))\| dr \\ &\leq k\|x(t) - x(s)\| + M_{\alpha'} \int_t^s \alpha(r) \theta(\|r\|) dr\end{aligned}$$

$$\leq k\|x(t)-x(s)\| + M_{\alpha'}\theta(\|r\|)\int_t^s \alpha(r)dr$$

$$\leq \frac{M_{\alpha'}\theta(\|r\|)}{1-k}\int_t^s \alpha(r)dr$$

$$\leq r_2$$

Accordingly,

$$\|x(t)-x(s)\| \leq r_2.$$

Consequently, $x \in M$.

Step 5: Now we show that there is an integer n_0 such that T is S-power-convex-condensing w.r.t. w about 0 and n_0, where w is the De Blasi measure of weak noncompactness. To see this, notice, for each bounded set $P \subseteq M$ and for each $t \in [0,T]$, that

$$W^{(1,0)}(T,S,P)(t) = \{x(t), x \in W^{(1,0)}(T,S,P)\}$$

$$\subseteq\{x(t) - f(x(t)) + x_0, x \in W^{(1,0)}(T,S,P)\}$$

$$+\{f(x(t)) - x_0, x \in W^{(1,0)}(T,S,P)\}$$

$$\subseteq T(P)(t) + \{f(x(t)) - f(0), x \in W^{(1,0)}(T,S,P)\}.$$

Thus

$$\gamma(W^{(1,0)}(T,S,P)(t)) \leq \gamma(T(P)(t)) + k\gamma\left(W^{(1,0)}(T,S,P)(t)\right) \tag{2.3}$$

Consequently,

$$\gamma(W^{(1,0)}(T,S,P)(t)) \leq \frac{1}{1-k}\gamma(T(P)(t))) \tag{2.4}$$

Further,

$$\gamma(T(P)(t)) = \gamma\left(\left\{f(0) + \int_0^t K(t,s)g(s,x(s))ds: x \in P\right\}\right)$$

$$\leq \gamma\left(t\overline{co}\{g(s,x(s)): x \in P, s \in [0,t]\}\right)$$

$$= t\gamma\left(\overline{co}\{g(s,x(s)): x \in P, s \in [0,t]\}\right)$$

$$\leq t\gamma\left(g([0,t] \times P[0,t])\right)$$

$$\leq t\lambda\gamma(P[0,t])$$

Theorem 1.1 implies (since M is equicontinuous) that

$$\gamma(T(P)(t)) \leq t\lambda\gamma(P) \tag{2.5}$$

Linking (2.4) and (2.5), we get

$$\gamma\big(W^{(1,0)}(T,S,P)(t) \leq \frac{t\lambda}{1-k}\gamma(P) \tag{2.6}$$

Using (2.4) we obtain,

$$\gamma\big(W^{(2,0)}(T,S,P)(t)\big) = \gamma(W^{(1,0)}(T,S,\overline{co}\big(W^{(1,0)}(T,S,P) \cup \{0\})(t)\big))$$

$$\leq \frac{1}{1-k}\gamma\left(T\left(\overline{co}\left(W^{(1,0)}(T,S,P) \cup \{0\}(t)\right)\right)\right).$$

Put $V = \overline{co}\big(W^{(1,0)}(T,S,P) \cup \{0\}\big)$. The use of (2.5) yields

$$\gamma\big(W^{(2,0)}(T,S,P)(t)\big) \leq \gamma(T(V)(t)) \leq \frac{1}{1-k}\gamma\left(\left\{x_0 + \int_0^t K(t,s)g\big(s,x(s)\big)ds \colon x \in V\right\}\right)$$

$$\leq \frac{1}{1-k}\gamma\left(\left\{\int_0^t g\big(s,x(s)\big)ds \colon x \in V\right\}\right).$$

Fix$t \in [0,T]$. We divide the interval $[0,t]$ into m parts $0 = t_0 < t_1 < \cdots < t_m = t$ in such a way that $\Delta t_i = t_i - t_{i-1} = \frac{t}{m}, i = 1, \ldots . m$.For each $\in V$, we have

$$\int_0^t g\big(s,x(s)\big)ds = \sum_{i=1}^m \int_{t_{i-1}}^{t_i} g\big(s,x(s)\big)ds \in \sum_{i=1}^m \Delta t_i \overline{co}\{g\big(s,x(s)\big) \colon x \in V, s \in [t_{i-1},t_i]\}$$

$$\subseteq \sum_{i=1}^m \Delta t_i \overline{co}\left(g\big([t_{i-1},t_i] \times V([t_{i-1},t_i])\big)\right).$$

Using again Theorem 1.1, we infer that for each $i = 2, \ldots, m$, there is an $s_i \in [t_{i-1}, t_i]$ such that

$$\sup_{s\in[t_{i-1},t_i]} \gamma(V(s)) = \gamma(V[t_{i-1},t_i]) = \gamma(V(s_i)) \tag{2.7}$$

Consequently,

$$\gamma\left(\left\{\int_0^t g\big(s,x(s)\big)ds \colon x \in V\right\}\right) \leq \sum_{i=}^m \Delta t_i \gamma\left(\overline{co}\left(g\big([t_{i-1},t_i] \times V([t_{i-1},t_i])\big)\right)\right)$$

$$\leq \lambda \sum_{i=1}^m \Delta t_i \gamma\left(\overline{co}\big(V([t_{i-1},t_i])\big)\right)$$

$$\leq \lambda \sum_{i=1}^m \Delta t_i \gamma(V(s_i))$$

On the other hand, if $m \to \infty$ then

$$\sum_{i=1}^m \Delta t_i \gamma(V(s_i)) \to \int_0^t \gamma(V(s))ds \tag{2.8}$$

Thus,

$$\gamma\left(\left\{\int_0^t g\big(s,x(s)\big)ds : x \in V\right\}\right) \le \int_0^t \gamma(V(s))ds \tag{2.9}$$

Using the regularity, the set additivity, the convex closure invariance of the De Blasi measure of weak noncompactness together with (2.5), we obtain

$$\gamma(V(s)) = \gamma\big(W^{(1,0)}(T,S,P)(s)\big) \le \frac{s\lambda}{1-k}\gamma(P) \tag{2.10}$$

and therefore

$$\int_0^t \gamma(V(s))ds \ \le \frac{\lambda}{(1-k)}\frac{t^2}{2}\gamma(P) \tag{2.11}$$

This implies

$$\gamma\big(W^{(2,0)}(T,S,P)(t)\big) \le \frac{(\lambda t)^2}{2(1-k)^2}\gamma(P) \tag{2.12}$$

By induction we get

$$\gamma\big(W^{(n,0)}(T,S,P)(t)\big) \le \frac{(\lambda t)^n}{n!(1-k)^n}\gamma(P) \tag{2.13}$$

Invoking Theorem 1.1 we obtain

$$\gamma\big(W^{(n,0)}(T,S,P)\big) \le \frac{(\lambda t)^n}{n!(1-k)^n}\gamma(P) \tag{2.14}$$

Since $\lim_{n\to\infty}\frac{(\lambda t)^n}{n!(1-k)^n} = 0$, then there is an n_0 with $\frac{(\lambda t)^{n_0}}{n_0!(1-k)^{n_0}} < 1$. This implies

$$\gamma\big(W^{(n_0,0)}(T,S,P)\big) < \gamma(P) \tag{2.15}$$

Finally $T$is sequentially weakly continuous. Let (x_n) be a sequence in $C([0,T],X)$ such that $x_n \to x$ for some $x \in C([0,T],X)$. By Theorem 1.2 we have $x_n(t) \to x(t)$ in X for all $t \in [0,T]$. By assumption (i) we have $g(s,x_n(s)) \to g(s,x(s))$ for all $s \in [0,T]$. The use of the Lebesgue dominated convergence theorem for Pettis integral gives $(Tx_n)(t) \to (Tx)(t)$ in for all $t \in [0,T]$. Using again Theorem 1.2, we obtain $Tx_n \to Tx$. Thus T is sequentially weakly continuous.

Applying Theorem 1.4, we get a fixed point for $S + T$ and hence a continuous solution to (3.1).

Conclusion

In this article we have discussed some theorems for the sum of two operators and used fixed point theorem named Darbo-Sadovskii to show that the solution will exist in the respective metric space. We have explained the measure of noncompactness also. Then we have applied the discussed result on Volterra-Hammerstin integral equation and showed that the solution exist in this integral equation.

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